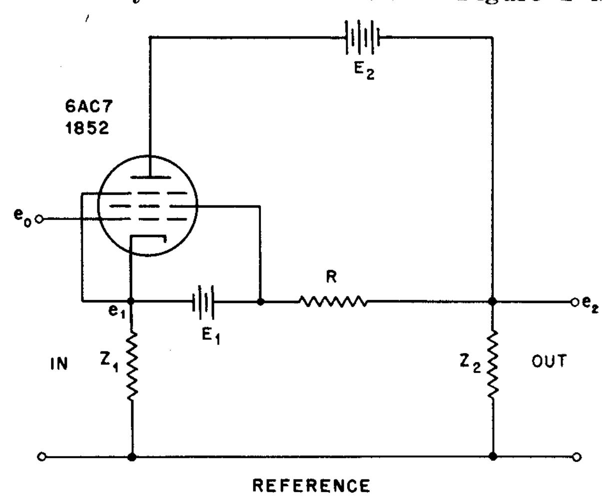
directly on the machine. Related variables could be plotted directly against one another, eliminating the time variable. Random disturbances, rather than simple periodic ones, could be imposed; this was revealing owing to their broad energy content over a wide band of frequencies. For this purpose ordinary tube noise was used. The writer naturally sought to apply similar techniques when, beginning in 1942, he met analogous problems in the design of aiming controls. That this applicability was not illusory is attested to by the several war projects in which useful ends have been served in dynamical studies by electronic simulation. Certain of the tricks employed earlier for regulatory studies have not been applied in the latter work, and the writer is rather eager to return to those studies, for servomechanism research and other applications, using the improved electric and electronic components more recently available. Somewhat more generally, it is felt that electronic simulative techniques will find growing and widespread usefulness for many recondite questions, both practical and academic, owing to the broad powers of representation embodied therein and to the speed with which exploratory manipulation may be performed in the laboratory.

4.3 FEEDBACK AMPLIFIERS [1,2,3,4,6]

The application of such amplifiers to simulative developments forms one of the most essential techniques in that art. Applied thus, the stability of the feedback circuit becomes of primary interest, leading to the remarkable situation in which stability studies as such are enhanced by the use of systems involving a set of feedback amplifiers, in the individual design of which such stability studies may be invaluable. But such a circumstance is not uncommon in research, where the talents of the detective, and dispassionate reasoning in general, find limitless opportunities.

It will be assumed here that the reader is familiar with the standard passive networks and with the corresponding dynamical systems which may be represented thereby without the necessity for feedback methods. Of the standard circuit elements only the resistor and the capacitor need be used in electronic simulation, and it is fortunate that these elements are available commercially in relatively pure, or "ideal," form.

There are two principal kinds of feedback in electronic circuits, which with their variations and mixtures have fundamental importance in simulative circuits. *Cathode feedback* and *plate feedback,* so called, comprise these two basic types, the names referring to the branches of the tube circuits in which retroactive operations are made to occur. Alternative terminology identifies these as *current feedback* and *voltage feedback,* respectively, in recognition of the electrical mode which acts most predominately in the two cases. In Figure 1 is



$$e_2 \doteq -\frac{Z_2(p)}{Z_1(p)} \cdot e_0$$

FIGURE 1. Cathode feedback circuit.

shown the fundamental cathode feedback circuit, or *cathode follower* adaptation, which was earlier employed by the writer in simulative assemblies. The inclusion of such material here does not imply that it carries classification. Much of the earlier work has been public knowledge, albeit not widespread. The writer had prepared and may publish descriptions of such work as applied to unclassified categories of engineering. A brief explanation should suffice. Since opposing currents circuits are produced in the impedance Z_1 by the sources E_1 and E_2, the voltage e_1 tends to follow the input voltage e_0 owing to the effect of the difference of these two voltages on the plate resistance of the tube. For proper choice of tube (preferably of high transconductance) and circuit constants, the error between e_0 and e_1 may be kept very small, provided of course that stable operation is attained. Thus e_0 may be "followed" very closely by e_1. Whereas e_0 may occur in a low-power-level circuit, and may be surrounded by high impedances, the very high grid resistance at which the present tube can be operated insures the relative noninterference with e_0 in its natural environment. Furthermore, power may be drawn from the circuit at e_1 since the feedback works to maintain this voltage in spite of applied loads, within considerable limits. Thus we have an important tool already, namely, an *isolating* or *buffer amplifier,* with unit gain. If now no current is drawn externally either from e_0 or e_2, and the sources E_1 and E_2 are exclusively involved in the present circuit, then the current in the impedances Z_1 and Z_2 must be equal. Thus $e_2/e_1 + Z_2/Z_1 = 0$, or approximately $e_2/e_0 = -Z_2/Z_1$. Now if e_2 is repeated by a subsequent buffer amplifier, a means is provided by the representation of many operational characteristics. If in particular the two impedances are simply equal resistors, a *polarity reverser* or *phase inverter,* so called, results. For integration with respect to time, Z_2 may be a capacitor and Z_1 a resistor. Conversely, for differentiation with respect to time, the roles of resistor and capacitor may be reversed. The generalizations which are possible may easily be imagined. Initial conditions are imposed in straightforward ways, although certain ingenious processes have become useful. In the above integrator, the lower limit of integration may be established at any time simply by imposing a momentary short across the capacitor. A typical simulator may comprise a number of such feedback components, interconnected either directly or through the appropriate passive networks. In this procedure, criteria such as that of *impedance-matching* may be forgotten. Addition, or the formation of linear combinations, for two or more voltages, and hence of the variables which these voltages represent, may be carried out in a number of ways with the basic circuit of Figure 1. For example, several such feedback circuits may be arranged to possess the impedance Z_2 in common. Or again, these voltages may be connected by a high-resistance dissipative network in such a way that e_0 is proportional to the desired sum or linear combination, although care must then be taken not to load the previous systems. For subtraction one may always phase invert an odd number of times. The only failing of the cathode feedback circuit, as shown, is the necessity for at least one "high" battery (E_1). The anode source (E_2), of course, must also be "high" unless Z_2 is zero, as it may be in the simple isolating amplifier, for example. Thus each such circuit must have at least one voltage source to itself. This requirement, however, has not been found too great a burden in the laboratory, since the drain may be kept quite small. The writer used a large bank of dry batteries for this purpose, in his own work, carefully shielded to prevent capacitive interstage coupling, and found it adequate to replace them every two years. Modifications are possible which will permit the adaptation of such circuits to single-source powering, but such sources must be meticulously regulated to present zero impedance and hence zero coupling between the using channels. Furthermore, such modifications complicate the circuits so much that plate feedback might as well be used. We now turn to plate feedback.

A common plate voltage source suffices for the plate feedback circuit, shown in Figure 2,

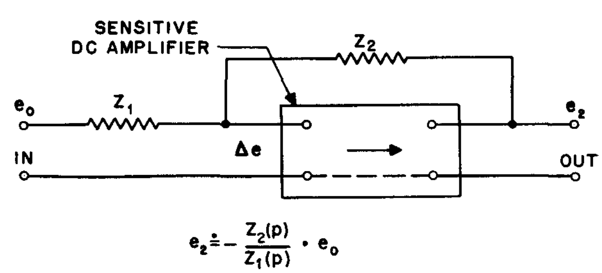


FIGURE 2. Plate feedback circuit.

if the high-gain d-c amplifier which is involved (see Figure 3) is properly designed. In this connection the term "high-gain" is employed in a relative sense only, as compared, for example, to the unity-gain *isolating* amplifier. (The gain figures for typical a-c amplifiers are not referred to in this context.) Only one such design is here shown. As suggested above, the supply sources should be well regulated. Referring to Figure 2, it should be remarked that the function of the amplifier shown there in the block is to convert the small balance voltage Δe into a corresponding but much expanded variation in the voltage e_2, and to reverse the sign. The effect is to keep the voltage Δe very near zero. As above in the current-feedback circuit, the point of operation (for the first tube) is so chosen that the grid input impedance is very high. Thus the currents in the impedances Z_1 and Z_2 are substantially equal. Hence as before, $e_2/e_0 \doteq -Z_2/Z_1$; and e_2 may be loaded within reasonable bounds. The operation of addition on several variables may be carried out, as in the second method mentioned above for the cathode feedback circuit, by substituting a number of resistive paths for Z_1, Z_2 being also resistive, and each such path initiating at one of the variable voltages to be summed or combined. The coefficients of combination are adjustable by the separate connecting resistances.

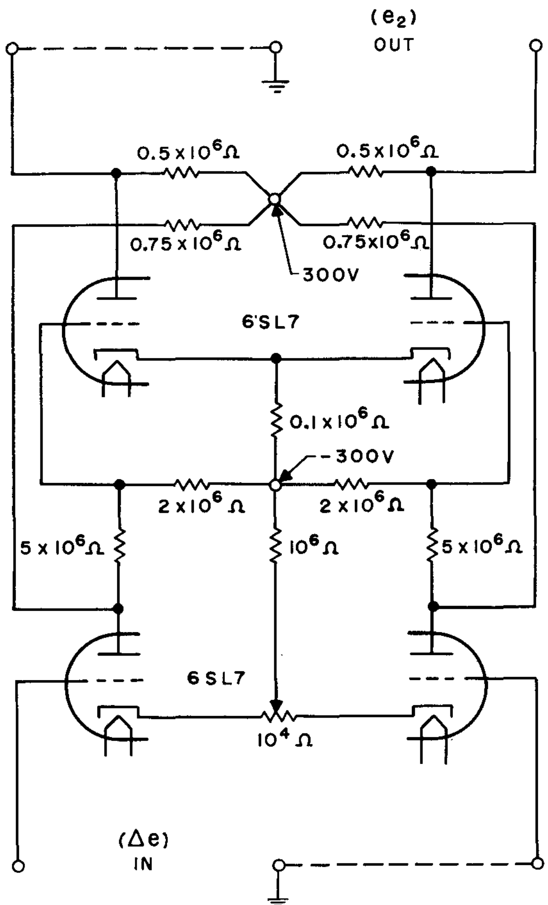


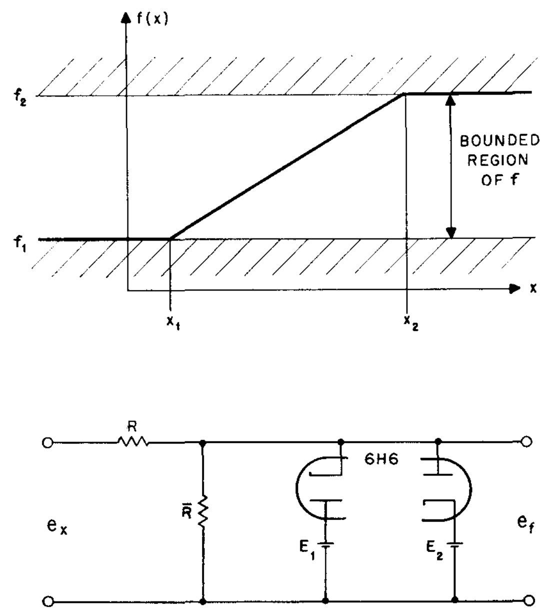
FIGURE 3. Type of amplifier indicated in Figure 2.

There are a certain number of feedback applications which do not fall into those types above classified. For example it is possible thus to provide circuits which have extremely long time constants, or long "lags" of low order, for special purposes. Such time constants are well in excess of those obtainable passively, even with the largest resistance and capacitance values available in the laboratory. Representation of the various kinds of servomechanisms or degenerative "telemeters" in prototype systems, as is to be expected, may be accomplished[1] quite straightforwardly by electronic feedback components in the process of simulative construction.

4.4 OTHER SIMULATIVE COMPONENTS AND COMBINATIONS

The amplifier circuits already referred to enable the experimenter to prepare, for his edification, extremely flexible models of dynamical systems. By simple interconnection, he may readily assemble the counterpart of any physical entity governed by a linear differential equation, of finite order, with constant coefficients. But he is not thus limited, although within this realm lie many significant dynamic relationships of sorts now submitted to analysis. For the larger class of linear relations represented by equations with coefficients which vary with the independent variable, here time, a programming of the adjustable parameters suffices. Similarly nonhomogeneous relations, with prescribed "forcing functions" (unfortunate phrase!), are representable. The greatest advantage of this kind of simulation, however, and the crux of its power, in relation to competitive methods, is the ability in simple manner to incorporate nonlinear dynamics. We need not speak of the analytic difficulties which are there involved, nor of the physical importance of nonlinear systems in general. Suffice it to say that the resulting problems preoccupy and harass some of the best available mathematical brains.

One very simple source of nonlinearity is the presence of a mere boundary in the range of a variable. Many examples may be cited, and may amply be supplied by the reader. Probably the simplest examples are furnished by transformations from a given variable to a function thereof which is either nonincreasing or nondecreasing over the entire range to be considered, and in which for at least certain portions of the range the function is unchanging. An instance is shown in Figure 4. The elec-



SPECIFICATIONS:
1. R LARGE COMPARED WITH "CLOSED" RESISTANCE OF DIODES
2. R AND $\bar{R}$ COMPARABLE WITH "OPEN" RESISTANCE OF DIODES

FIGURE 4. Boundary function and its simulative counterpart.

tronic embodiment of such a transformation is simple. It may involve only a pair of rectifying diodes, as contained for example in a 6H6 tube, so biased that voltages corresponding to variation of the first variable beyond the bounded interval produce no further change in the function of, or the transformed version of, that variable. Such an arrangement, shown also in Figure 4, must always be approximate, but the approximation may be refined to any desired degree by appropriate choice of the electrical circumstances. Only one type of nonlinear component is thus illustrated; it is evident, however, that by combination of such circuits a great variety of similar, and more intricate, relations may be embodied. Nor are we restricted to functional relations representable graphically by a single curve, or analytically by single-valued dependences. Such physical relationships as *inertness,* when for example there is *backlash* or *play* in a mechanical connection, may be simulated through the application of biased rectifiers in circuits where charge may be stored over reasonable intervals. In general, the inclusion of major nonlinearities of this sort alter fundamentally the stability considerations in control systems or in any apparatus where the dynamic performance is of interest. It is thus of great importance to be able to treat such nonlinearities without the endless drudgery which is essential when straightforward analysis is the only recourse, however otherwise educational or character-building such a process may be for the performer.

A useful tool in simulation, for that general class of dynamics, particularly, in which nonlinear relations are included (it might have been more appropriate earlier to have defined nonlinear systems as those in which the *additivity* or *superposition* principle does not apply), is the follower or servomechanism which transforms voltage into a corresponding resistance. Consider first the case in which proportional correspondence is enforced. Many such components have been built and applied in the simulative ventures we have dealt in. The usefulness of such a properly made component, which is great, is not at all limited to the representation of nonlinearities, but appears wherever the automatic manipulation of a resistor or potentiometer may be desired. The construction may be very neat and simple, and high speed and precision has been shown fairly easy to attain. Naturally the time constants of the response should be well below those which are involved, purposely, in the remainder of the simulative channel in which such a component occurs; it has been satisfactory to employ pairs of circular, wire-wound potentiometers driven by small d-c motors energized either from polarized relays or from vacuum tube circuits. The error signal, which is supplied to an interpreting network and thence to the motor circuit, is derived both from the input voltage and from the voltage division of one of the driven potentiometers. The other potentiometer, driven for example in tandem with the first, provides a free resistor having resistance proportional, say, to the input voltage. Before turning to some applications of the proportional form, we merely mention that the introduction of any appropriate function may be accomplished, through the voltage-to-resistance follower, by embodiment of such function in the resistance versus motion law of the final resistor, or of its reciprocal or inverse in that of the follower-resistor itself.

A means for rapid conversion of a voltage into a proportional resistance enables the realization of many flights of fancy. We record only the soberer of these possibilities. It will be evident that certain of these may also be effected by logically simpler means, but we note that there is large advantage, for such purposes, in the repeated application of a given type of basic component.

Having a feedback amplifier, of either type above described, with the associated pair of impedances restricted to a pair of resistances, then it is possible, by making either such resistance proportional to a separate external voltage, to perform multiplication or division among voltages in various ways. The resulting product or quotient has also the form of a voltage. While those variables which are reflected in the resistance of either resistor pair cannot normally be made to take on negative values, or in certain cases even zero, modifications are possible whereby equivalent variation can be permitted. A general case is given by that in which the input voltage of the feedback amplifier is employed together with those which, respectively, set proportionally the "input" and "output" resistances thereof. Here the output voltage is proportional, through any desired factor of scale, to the ratio of the product of the first two voltages to the third voltage. Simple products or ratios, as well as reciprocals, may then be evaluated in a single component. Through duplicate roles, for example, a voltage, or rather the variable thereby repsented, may be squared through multiplication into itself. By including such an arrangement within a more comprehensive feedback loop, inverse operations such as the extraction of the square root may be performed. Out of necessity, we are speaking here only of *real* variables. With combinations of similar components, in series or cascade, it is evident that such delicate operations may be carried out as the raising of a variable to any fixed rational power. This type of conversion, of course, may be accompanied by any (otherwise) linear dynamic operation of the classes already mentioned.

Consider now integration with respect to variables other than time, or in general with respect to an arbitrary independent variable. An arrangement was proposed[b] for this purpose, but was not constructed practically for a variety of reasons, principally because no need persisted for it which was not more easily satisfied by other means. Consider a time integrator as described, with fixed resistive and capacitive elements, and let it be supplied as input with a voltage which is the product, obtained as already indicated, of a primary voltage e_1 (preferably, in this case, always of one sign) and the time derivative, evaluated in the regular feedback manner, of a second input voltage e_2. The output voltage e_3 of the time integrator is thus the integral of the voltage e_1 with respect to the voltage e_2. Thus

$$e_3 = \frac{1}{RC}\int_{t_0}^{t} e_1 \frac{de_2}{dt}\,dt = \frac{1}{RC}\int_{e_1(t_0)}^{e_1} e_1\,de_2\,.$$

Within certain limitations, of which it is not difficult to imagine the removal, and with precision of the order of 0.001 to 0.01, it is evident that this more general type of integration is feasible, and the potentialities of electronic simulation are thus extended to the dynamical realms of the better-known *differential analyzer*. Much practical development, however, remains to be completed along these lines, although only the fundamental elements already treated need be involved.

Returning momentarily to linear systems, with the continued implication none the less of nonlinear generalizations and ramifications,

[b] We should like to acknowledge the collaboration of Loebe Julie in connection with this item, and also with certain other conjectural plans for electronic simulative measures which are referred to here.